Exact Solution of a Charge-Asymmetric Two-Dimensional Coulomb Gas

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Abstract

The model under consideration is an asymmetric two-dimensional Coulomb gas of positively ($q_1 = +1$) and negatively ($q_2 = -1/2$) charged pointlike particles, interacting via a logarithmic potential. This continuous system is stable against collapse of positive-negative pairs of charges for the dimensionless coupling constant (inverse temperature) $\beta < 4$. The mapping of the Coulomb gas is made onto the complex Bullough-Dodd model, and recent results about that integrable 2D field theory are used. The mapping provides the full thermodynamics (the free energy, the internal energy, the specific heat) and the large-distance asymptotics of the particle correlation functions, in the whole stability regime of the plasma. The results are checked by a small-β expansion and close to the collapse $\beta = 4$ point. The comparison is made with the exactly solvable symmetric version of the model ($q_1 = +1, q_2 = -1$), and some fundamental changes in statistics caused by the charge asymmetry are pointed out.

KEY WORDS: Two-dimensional Coulomb gas; exactly solvable models; Bullough-Dodd model; thermodynamics; pair correlation functions.

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1 Introduction

A two-dimensional (2D) Coulomb plasma is the continuous system of charged particles, plus perhaps a uniformly charged neutralizing background, confined to a plane and interacting via the logarithmic Coulomb potential. In this paper, the classical equilibrium statistical mechanics at the (dimensionless) inverse temperature β is studied. We will restrict ourselves to the simple point-particle Coulomb systems: if there are at least two species of charged particles with opposite signs, the Coulomb system is stable against the charge collapse for small enough $\beta < \beta_{\rm col}$ (in 3D, the collapse point $\beta_{\rm col} \to 0$ and the stability requires quantum mechanics [1]). Going beyond the collapse point needs a short-distance (e.g. hard core) regularization of the Coulomb interaction. The famous Kosterlitz-Thouless (KT) transition [2] of infinite order from a high-temperature conducting phase to a low-temperature insulating phase takes place at some $\beta_{\rm KT} > \beta_{\rm col}$. Like in other dimensions, the 2D Coulomb systems admit the Debye-Hückel treatment in the high-temperature limit $\beta \to 0$, as was rigorously proven in Ref. [3]. As concerns the bulk thermodynamics, based on the scale invariance of the logarithmic potential, the density derivatives of the free energy, like the pressure, can be obtained exactly in the whole stability range of temperatures [4]-[6]. On the other hand, the temperature derivatives of the free energy, like the internal energy or the specific heat, are nontrivial statistical quantities which cannot be obtained by simple means and they are known only in special cases (see below).

The most studied versions of the Coulomb plasma are the one-component plasma (OCP) and the symmetric two-component plasma (TCP), or Coulomb gas.

The 2D OCP of equally, say unit, charged pointlike particles in a uniform neutralizing background is formally related to the fractional quantum Hall effect [7, 8]. No collapse occurs. There are indications that, around $\beta \sim 142$, the fluid system undergoes a phase transition to a 2D Wigner crystal [9]. In a more recent paper [10], the existence of this transition has been put in doubt. The model is exactly solvable at $\beta = 2$ by mapping onto

free fermions, in the bulk [11] as well as in some inhomogeneous situations (see review [12]).

The symmetric 2D TCP consists of oppositely ± 1 charged particles, no background is present. The collapse point $\beta_{\rm col} = 2$ coincides with the exactly solvable free-fermion point of the equivalent Thirring model [13]-[15]. At this collapse point, for a fixed fugacity, while the free energy diverges, truncated Ursell functions are nonzero and finite. Quite recently [16], the complete bulk thermodynamics of the symmetric 2D TCP was derived exactly in the whole stability region $\beta < 2$. The mapping onto a bulk 2D sine-Gordon theory with a conformal normalization of the cos-field was made, and recent results about that field theory were applied. Subsequently, the surface tension of the same model in contact with an ideal-conductor [17] and with an ideal-dielectric [18] rectilinear walls was obtained via a mapping onto integrable 2D boundary sine-Gordon theories with Dirichlet and Neumann boundary conditions, respectively. The large-distance behavior of the bulk charge [19] and density [20] correlation functions was derived exactly by exploring the form-factor theory of the equivalent sine-Gordon model.

Hansen and Viot [21] have introduced a general asymmetric 2D Coulomb gas which consists of two species of pointlike particles with positive and negative charges of arbitrary strengths q_σ ($\sigma = 1, 2$). For $|q_1/q_2|$ being an integer, the model describes a plasma of electrons and ions of integer valence. Without any loss of generality, we choose

$$q_1 = +1 \quad \text{and} \quad q_2 = -1/Q \tag{1.1}$$

where $Q = 1, 2, \dots$ is a positive integer. The model interpolates between the symmetric TCP ($Q = 1$) and, after subtracting the kinetic energy of 2-species, the OCP obtained as the extreme asymmetry case $Q \to \infty$. The Boltzmann factor of a pair of positive and negative charges, $r^{-\beta/Q}$, is integrable at short distance in 2D space for $\beta < 2Q$, so the collapse point is $\beta_{\rm col} = 2Q$ [21]. In the case of a vanishing but nonzero hard core around particles, the KT phase transition was conjectured to take place at $\beta_{\rm KT} = 4Q$ [22]. Highly asymmetric Coulomb mixtures in the strong-coupling regime have attracted

much attention in the last years [23]-[26] due to the phenomenon of overcharging (charge inversion), i.e. the situation when the number of counterions in the vicinity of a macroion is so high that the macro-charge is overcompensated.

In this work, we report the exact solution of the $Q = 2$ asymmetric 2D Coulomb gas, with the lowest degree of charge asymmetry $q_1 = 1$ and $q_2 = -1/2$, in the whole stability interval of pointlike particles $\beta < 4$. The exact solution involves the complete bulk thermodynamics (the free energy, the internal energy, the specific heat) and the large-distance behavior of the particle correlation functions. It is obtained by mapping the underlying Coulomb gas onto a member of 2D Toda field theories [27], namely the complex Bullough-Dodd (cBD) model [28]-[30] with a conformal normalization of the exponential field. Recent results about that integrable field theory, derived by using the Thermodynamic Bethe Ansatz (thermodynamics) and the form-factor method (correlation functions), are applied. Since the calculations in the cBD model are based on special analyticity assumptions, which are not yet rigorously proven, we check the results for the plasma by a small-β expansion, using a renormalized Mayer expansion in density [31, 32], and close to the collapse $\beta = 4$ point, using an electroneutrality sum rule [33] combined with an independent-pair conjecture made by Hauge and Hemmer [6]. The comparison is made with the symmetric version of the model, and the fundamental changes in statistics due to the charge asymmetry are pointed out.

The paper is organized as follows. In Section 2, all relevant aspects of the mapping between the asymmetric $1/-\frac{1}{2}$ 2D Coulomb gas and the cBD model are presented. The complete thermodynamics of the Coulomb system is derived in Section 3. The asymptotic large-distance behavior of the particle correlation functions is obtained within the form-factor method in Section 4. Section 5 is a brief recapitulation with some concluding remarks. The results are checked by the small-β expansions in Appendix A and close to the collapse $\beta = 4$ point in Appendix B.

2 Field-theoretical representation of the asymmetric 2D Coulomb gas

We consider the asymmetric TCP made up of two species of pointlike particles with charges q_σ ($\sigma = 1, 2$) given by (1.1). The particles are confined to an infinite 2D space of points $\mathbf{r} \in R^2$, and interact with each other by the pair Coulomb potential. The Coulomb potential v at spatial position $\mathbf{r}$, induced by a unit charge at the origin, is given by the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}) \tag{2.1}$$

as follows

$$v(\mathbf{r}) = -\ln\left(|\mathbf{r}|/r_0\right) \tag{2.2}$$

The length constant r_0, which fixes the zero point of the Coulomb potential, will be set for simplicity to unity. The interaction energy E of particles $\{j\}$ reads

$$E = \sum_{j<k} q_{\sigma_j} q_{\sigma_k} v(|\mathbf{r}_j - \mathbf{r}_k|) \tag{2.3}$$

Introducing the microscopic density of particles of species σ, $\hat{n}_\sigma(\mathbf{r}) = \sum_j \delta_{\sigma,\sigma_j}\delta(\mathbf{r} - \mathbf{r}_j)$, the microscopic densities of the total particle number and of the charge are

$$\hat{n}(\mathbf{r}) = \sum_\sigma \hat{n}_\sigma(\mathbf{r}), \qquad \hat{\rho}(\mathbf{r}) = \sum_\sigma q_\sigma \hat{n}_\sigma(\mathbf{r}) \tag{2.4}$$

respectively. The energy (2.3) can be thus written as

$$E = \frac{1}{2}\int \mathrm{d}^2 r \int \mathrm{d}^2 r' \hat{\rho}(\mathbf{r}) v(|\mathbf{r} - \mathbf{r}'|)\hat{\rho}(\mathbf{r}') - \frac{1}{2} v(0) \sum_j q_{\sigma_j}^2 \tag{2.5}$$

where $v(0)$ is the (divergent) self-energy.

We will work in the grand canonical ensemble, with position-dependent fugacities $z_\sigma(\mathbf{r})$ of species $\sigma = 1, 2$. The grand partition function Ξ at inverse temperature β, considered as a functional of the species fugacities, is defined by

$$\begin{aligned} \Xi[z_1, z_2] \quad &= \quad \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \frac{1}{N_1!} \frac{1}{N_2!} \int \prod_{j=1}^{N_1} \left[\mathrm{d}^2 u_j z_1(\mathbf{u}_j)\right] \\ &\quad \times \prod_{j=1}^{N_2} \left[\mathrm{d}^2 v_j z_2(\mathbf{v}_j)\right] \exp\left[-\beta E_{N_1,N_2}(u, v)\right] \end{aligned} \tag{2.6}$$

where $E_{N_1,N_2}(u,v)$ denotes the Coulomb interaction energy (2.3), resp. (2.5), of N_1 particles of type 1 at vector positions $\{\mathbf{u}_j\}_{j=1}^{N_1}$ and N_2 particles of type 2 at positions $\{\mathbf{v}_j\}_{j=1}^{N_2}$. The statistical quantity Ξ can be expressed in terms of a 2D Euclidean theory with the aid of the standard procedure (see, e.g., ref. [35]). Using the representation (2.5) for E in $\exp(-\beta E)$ and assuming that $-\Delta/(2\pi)$ is the inverse operator of $v(\mathbf{r})$ [see relation (2.1)], one applies the Hubbard-Stratonovich transformation

$$\exp\left[-\frac{\beta}{2}\int \mathrm{d}^2 r \int \mathrm{d}^2 r' \hat{\rho}(\mathbf{r}) v(|\mathbf{r}-\mathbf{r}'|)\hat{\rho}(\mathbf{r}')\right] = \frac{\int \mathcal{D}\phi \exp\left[\int \mathrm{d}^2 r \left(\frac{1}{16\pi}\phi\Delta\phi + \mathrm{i}\sqrt{\beta/4}\ \phi\hat{\rho}\right)\right]}{\int \mathcal{D}\phi \exp\left(\int \mathrm{d}^2 r \frac{1}{16\pi}\phi\Delta\phi\right)} \tag{2.7}$$

where $\phi(\mathbf{r})$ is a real scalar field and $\int \mathcal{D}\phi$ denotes the functional integration over this field. The term $\phi\Delta\phi$ can be turned into $-(\nabla\phi)^2$ by using integration by parts. The summation over N_1 and N_2 in (2.6) then implies

$$\Xi[z_1, z_2] = \frac{\int \mathcal{D}\phi \exp\left(-S[z_1, z_2]\right)}{\int \mathcal{D}\phi \exp\left(-S[0,0]\right)} \tag{2.8}$$

where the action of the 2D Euclidean field theory takes the form

$$S[z_1, z_2] = \int \mathrm{d}^2 r \left[\frac{1}{16\pi}(\nabla\phi)^2 - z_1(\mathbf{r})\mathrm{e}^{\mathrm{i}b\phi} - z_2(\mathbf{r})\mathrm{e}^{-\mathrm{i}(b/Q)\phi}\right] \tag{2.9a}$$

$$b^2 = \beta/4 \tag{2.9b}$$

Here, the fugacities z_σ ($\sigma = 1,2$) are renormalized by the self-energy terms $\exp[\beta v(0) q_\sigma^2/2]$. For the homogeneous system with uniform species fugacities, $z_\sigma(\mathbf{r}) = z_\sigma$, the action (2.9) simplifies to

$$S(z_1, z_2) = \int \mathrm{d}^2 r \left[\frac{1}{16\pi}(\nabla\phi)^2 - z_1\mathrm{e}^{\mathrm{i}b\phi} - z_2\mathrm{e}^{-\mathrm{i}(b/Q)\phi}\right] \tag{2.10}$$

Note that this action is complex, except in the symmetric case $Q = 1$ with $z_1 = z_2$.

The field representation of the multi-particle densities can be obtained from the functional generator Ξ, defined by (2.8) and (2.9), in a straightforward way. At the one-particle level, the density of particles of type $\sigma(= 1,2)$ is given by

$$\begin{aligned} n_\sigma &= \langle \hat{n}_\sigma(\mathbf{r})\rangle \\ &= z_\sigma \frac{1}{\Xi}\frac{\delta\Xi}{\delta z_\sigma(\mathbf{r})}\Big|_{\mathrm{uniform}} \end{aligned} \tag{2.11}$$

Consequently,

$$n_\sigma = z_\sigma \langle \mathrm{e}^{\mathrm{i}bq_\sigma \phi} \rangle \tag{2.12}$$

where $\langle \cdots \rangle$ denotes the averaging over the action (2.10). Note that under the shift of the field variable $\phi \to \phi + \phi_0$ in (2.10) (which has no effect on Ξ as the whole), the species fugacities change as $z_1 \to z_1 \exp(\mathrm{i}b\phi_0)$ and $z_2 \to z_2 \exp[-\mathrm{i}(b/Q)\phi_0]$, and therefore Ξ depends only on the combination $z_1(z_2)^Q$. According to the definition (2.11), this property of Ξ implies the neutrality condition $n_1 - (1/Q)n_2 = 0$. At the two-particle level, one introduces the two-body densities

$$\begin{aligned} n_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') &= \langle \hat{n}_\sigma(\mathbf{r}) \hat{n}_{\sigma'}(\mathbf{r}') \rangle - n_\sigma \delta_{\sigma\sigma'} \delta(\mathbf{r} - \mathbf{r}') \\ &= z_\sigma z_{\sigma'} \frac{1}{\Xi} \frac{\delta^2 \Xi}{\delta z_\sigma(\mathbf{r}) \delta z_{\sigma'}(\mathbf{r}')} \Big|_{\text{uniform}} \end{aligned} \tag{2.13}$$

so that

$$n_{\sigma\sigma'}(|\mathbf{r} - \mathbf{r}'|) = z_\sigma z_{\sigma'} \langle \mathrm{e}^{\mathrm{i}bq_\sigma \phi(\mathbf{r})} \mathrm{e}^{\mathrm{i}bq_{\sigma'} \phi(\mathbf{r}')} \rangle \tag{2.14}$$

It is useful to introduce also the pair distribution functions

$$g_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') = \frac{n_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}')}{n_\sigma n_{\sigma'}} \tag{2.15}$$

and the (truncated) pair correlation functions

$$h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') = g_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') - 1 \tag{2.16}$$

In statistical mechanics, the short-distance behavior of the pair distribution function is dominated by the Boltzmann factor of the pair Coulomb potential [21, 34],

$$g_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \sim C_{\sigma\sigma'} |\mathbf{r} - \mathbf{r}'|^{\beta q_\sigma q_{\sigma'}} \quad \text{as} \quad |\mathbf{r} - \mathbf{r}'| \to 0 \tag{2.17}$$

(provided β is small enough as explained below). The prefactors $C_{\sigma\sigma'}$ are expressible as follows

$$C_{\sigma\sigma'} = \exp\left[\beta\left(\mu^{\text{ex}}_{q_\sigma} + \mu^{\text{ex}}_{q_{\sigma'}} - \mu^{\text{ex}}_{q_\sigma + q_{\sigma'}}\right)\right] \tag{2.18}$$

where $\mu^{\text{ex}}_{q_\sigma}$ is the excess chemical potential of species σ defined by

$$\beta \mu^{\text{ex}}_{q_\sigma} = \ln\left(\frac{z_\sigma}{n_\sigma}\right) \tag{2.19}$$

and μ_q^{ex} with q arbitrary represents an extended definition of the excess chemical potential for a "guest" particle of charge q put into the considered Coulomb gas (1.1). To get the stability region of μ_q^{ex}, one has to consider the interaction Boltzmann factor of the q-charge with an opposite charge from the plasma. When $q > 0$ ($q < 0$), the Boltzmann factor with the opposite $-1/Q$ ($+1$) plasma charge at distance r, $r^{-\beta q/Q}$ ($r^{\beta q}$), is integrable at small r if and only if $\beta < 2Q/q$ ($\beta < -2/q$). μ_q^{ex} is therefore finite if $\beta < (Q+1)/|q| + (Q-1)/q$ and goes to $-\infty$ outside of this stability region. According to the formula (2.18) this means that the prefactor C_{12} remains finite in the whole stability region $\beta < 2Q$, while the prefactors C_{11} and C_{22} are finite only if $\beta < Q$. The divergence of C_{11} and C_{22} in the middle of the stability region, at the point $\beta = Q$, is accompanied by a change in the short-distance behavior (2.17) of g_{11} and g_{22}. The general analysis in Ref. [21] shows that, in the short-distance limit $r \to 0$,

$$
\begin{aligned}
g_{11}(r) &\propto r^{\beta}, \qquad \beta < Q \\
&\propto r^{2m+\beta[1-m(4Q+1-m)/(2Q^2)]}, \qquad \frac{2Q^2}{2Q-m+1} < \beta < \frac{2Q^2}{2Q-m}
\end{aligned}
\tag{2.20a}
$$

where m is an integer from the interval $1 \le m \le Q$, and

$$
\begin{aligned}
g_{22}(r) &\propto r^{\beta/Q^2}, \qquad \beta < Q \\
&\propto r^{2-\beta(2Q-1)/Q^2}, \qquad Q < \beta < 2Q
\end{aligned}
\tag{2.20b}
$$

Close to the collapse point $\beta \to 2Q$, one finds

$$g_{11}(r) \propto r^{Q-1} \tag{2.21a}$$

$$g_{22}(r) \propto r^{(2/Q)-2} \tag{2.21b}$$

The collapse relation (2.21b) tells us that, for $Q > 1$, $g_{22}(r)$ does not vanish as $r \to 0$ (as is intuitively expected since equally charged particles repel each other), but diverges. This is related to a paradoxical clustering of counterions in the asymmetric plasma.

We now intend to derive the short-distance behavior (2.17), say for g_{11}, in terms of

the 2D field theory (2.10). According to the definitions (2.6), (2.13) and (2.15), we have

$$g_{11}(\mathbf{r},\mathbf{r}') = |\mathbf{r}-\mathbf{r}'|^{\beta}\left(\frac{z_1}{n_1}\right)^2 \frac{1}{\Xi}\sum_{N_1=2}^{\infty}\sum_{N_2=0}^{\infty}\frac{N_1(N_1-1)}{N_1!N_2!}\int\prod_{j=1}^{N_1-2}\left[\mathrm{d}^2u_j z_1(\mathbf{u}_j)\right]\prod_{j=1}^{N_2}\left[\mathrm{d}^2v_j z_2(\mathbf{v}_j)\right]$$
$$\times\exp\left\{-\beta E_{N_1-2,N_2}(u,v)-\beta\int\mathrm{d}^2r''\left[v(\mathbf{r}-\mathbf{r}'')+v(\mathbf{r}'-\mathbf{r}'')\right]\hat{\rho}(\mathbf{r}'')\right\} \tag{2.22}$$

where $\hat{\rho}$ denotes the microscopic charge density of N_1-2 particles of type 1 and N_2 particles of type 2. One shifts $N_1 \to N_1-2$ in the summation over N_1. When $\mathbf{r}'\to\mathbf{r}$, the exponential in (2.22) can be written in the form presented on the lhs of Eq. (2.7) with the substitution $\hat{\rho}(\mathbf{r}'')\to\hat{\rho}(\mathbf{r}'')+2\delta(\mathbf{r}''-\mathbf{r})$. Using the Hubbard-Stratonovich identity (2.7), the previously generated single term $\mathrm{i}\sqrt{\beta/4}\,\phi\hat{\rho}$ now involves two terms: $\mathrm{i}\sqrt{\beta/4}\,\phi\hat{\rho}+\mathrm{i}2\sqrt{\beta/4}\phi(\mathbf{r})$. One integrates over particle coordinates, except the fixed $\mathbf{r}$-coordinate, and ends up with

$$g_{11}(|\mathbf{r}-\mathbf{r}'|)\sim|\mathbf{r}-\mathbf{r}'|^{\beta}\left(\frac{z_1}{n_1}\right)^2\langle\mathrm{e}^{\mathrm{i}2b\phi}\rangle \quad \text{as} \quad |\mathbf{r}-\mathbf{r}'|\to 0 \tag{2.23a}$$

Analogously, one finds that

$$g_{12}(|\mathbf{r}-\mathbf{r}'|) \sim |\mathbf{r}-\mathbf{r}'|^{-\beta/Q}\left(\frac{z_1}{n_1}\right)\left(\frac{z_2}{n_2}\right)\langle\mathrm{e}^{\mathrm{i}(1-1/Q)b\phi}\rangle \quad \text{as} \quad |\mathbf{r}-\mathbf{r}'|\to 0 \tag{2.23b}$$
$$g_{22}(|\mathbf{r}-\mathbf{r}'|) \sim |\mathbf{r}-\mathbf{r}'|^{\beta/Q^2}\left(\frac{z_2}{n_2}\right)^2\langle\mathrm{e}^{-\mathrm{i}(2b/Q)\phi}\rangle \quad \text{as} \quad |\mathbf{r}-\mathbf{r}'|\to 0 \tag{2.23c}$$

Comparing these relations with Eqs. (2.17) - (2.19), the excess chemical potential of a particle with charge q embedded into the considered plasma is related to the one-point expectation of the exponential field as follows

$$\exp\left(-\beta\mu_q^{\mathrm{ex}}\right)=\langle\mathrm{e}^{\mathrm{i}qb\phi}\rangle \tag{2.24}$$

Note that relations (2.12) are special cases of the general formula (2.24). According to the analysis after Eq. (2.19) and with respect to the relationship (2.9b), the one-point expectation $\langle\mathrm{e}^{\mathrm{i}qb\phi}\rangle$ is finite if $4b^2<(Q+1)/|q|+(Q-1)/q$, and goes to $+\infty$ otherwise. Comparing relations (2.23) with Eqs. (2.14) and (2.15), the short-distance behavior of two-point expectations of the exponential field is given by

$$\langle\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\mathrm{e}^{\mathrm{i}q'b\phi(\mathbf{r}')}\rangle\sim|\mathbf{r}-\mathbf{r}'|^{4b^2qq'}\langle\mathrm{e}^{\mathrm{i}(q+q')b\phi}\rangle \quad \text{as} \quad |\mathbf{r}-\mathbf{r}'|\to 0 \tag{2.25}$$

In quantum field theory, the model (2.10) can be regarded as the Gaussian conformal field theory perturbed by the operator $-z_1 \exp(\mathrm{i}b\phi) - z_2 \exp[-\mathrm{i}(b/Q)\phi]$. The species fugacities z_1 and z_2 are renormalized by the (divergent) self-energy factors. To give the parameters z_1 and z_2 a precise meaning, one has to fix the normalization of the adjoint exponential fields. The normalization which corresponds to the short-distance limit of the two-point function (2.25) is known as conformal. Under the conformal normalization, the diverging self-energy factor disappears from statistical relationships. This makes the bridge between the underlying asymmetric Coulomb plasma and the corresponding field theory with the action (2.10), treated within the Conformal Perturbation theory.

Note that q and q' should be sufficiently small to ensure that (2.25) is the leading short-distance asymptotics. From the point of view of the above equivalence, the crucial is the stability of the two-point expectation asymptotics (2.25) for two opposite charges $q = 1$ and $q' = -1/Q$. In that case, the stability region of $\langle \mathrm{e}^{\mathrm{i}(q+q')b\phi} \rangle$ is $\beta < \beta_{\mathrm{stab}} = 2Q^2/(Q-1)$; for the symmetric Coulomb gas $\beta_{\mathrm{stab}} \to \infty$, for $Q = 2$ β_{stab} incidently coincides with $\beta_{\mathrm{KT}} = 8$ and for any finite Q it holds $\beta_{\mathrm{stab}} > \beta_{\mathrm{col}}$.

3 Thermodynamics of an asymmetric Coulomb gas

There is a large class of integrable 2D field theories, known as the affine Toda theories, which are based on the Dynkin-diagram classification of simple Lie groups (for a nice review, see Ref. [27]). Let $\mathcal{G}$ be a simple Lie algebra of rank r, $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_r\}$ a set of simple r-dimensional roots of the corresponding Dynkin diagram and $-\boldsymbol{e} = \sum_{i=1}^r n_i \boldsymbol{e}_i$ (the coefficients $\{n_i\}$ are called Kac labels) the maximal root. The affine Toda theory built on $\mathcal{G}$ is defined by the action

$$S = \int \mathrm{d}^2 r \left[\frac{1}{16\pi} (\partial_\mu \boldsymbol{\phi})^2 + \sum_{i=1}^r z_i \mathrm{e}^{b\boldsymbol{e}_i \cdot \boldsymbol{\phi}} + z_{r+1} \mathrm{e}^{b\boldsymbol{e} \cdot \boldsymbol{\phi}} \right] \tag{3.1}$$

where the field $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_r)$ consists of r real scalar components and b is the real coupling constant. The fields in Eq. (3.1) are normalized so that at $\{z_i = 0\}_{i=1}^{r+1}$

$$\langle \phi_a(\mathbf{r}) \phi_b(\mathbf{r}') \rangle = -4\delta_{ab} \ln |\mathbf{r} - \mathbf{r}'| \tag{3.2}$$

For the one-component ($r = 1$) case of interest, there exist two integrable Toda theories. When $\mathcal{G} = A_1^{(1)}$ Lie group with $e_1 = 1$ and $e = -1$, one obtains the sinh-Gordon (b real) or sine-Gordon (b pure imaginary) models. The sine-Gordon model is identified with the action (2.10) with $Q = 1$ and describes the thermodynamics of the symmetric 2D TCP (see Refs. [16]-[19]). When $\mathcal{G} = A_2^{(2)}$ Lie group with $e_1 = 1$ and $e = -1/2$, the action (3.1) takes the form

$$S_{\mathrm{BD}} = \int \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla \phi)^2 + z_1 \mathrm{e}^{b\phi} + z_2 \mathrm{e}^{-b\phi/2} \right] \tag{3.3}$$

This theory is known as the Bullough-Dodd (BD) model [28]. Its complex version, obtained via the substitutions $b \to \mathrm{i}b$ and $z_{1,2} \to -z_{1,2}$,

$$S_{\mathrm{cBD}} = \int \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla \phi)^2 - z_1 \mathrm{e}^{\mathrm{i}b\phi} - z_2 \mathrm{e}^{-\mathrm{i}(b/2)\phi} \right] \tag{3.4}$$

is referred to as the Zhiber-Mikhailov-Shabat model [29, 30], or simply the complex Bullough-Dodd (cBD) model. Comparing (3.4) with (2.10) one observes that the cBD model with the short-distance normalization (2.25) is the 2D field realization of the $Q = 2$ asymmetric TCP, i.e. the system of $+1$ and $-1/2$ charged particles. According to (2.9), the coupling constant b is related to the inverse temperature by $b^2 = \beta/4$ and the parameters z_σ ($\sigma = 1, 2$) represent the renormalized species fugacities.

The particle spectrum of the BD model (3.3) consists of a single neutral particle of mass m. The two-particle S-matrix was described in Ref. [36]. The (dimensionless) specific grand potential

$$-\omega = \lim_{V \to \infty} \frac{1}{V} \ln \Xi \tag{3.5}$$

where Ξ is given by (2.8) and V is the volume, was obtained in terms of the particle mass m in Ref. [37] following the Thermodynamic Bethe Ansatz technique [38, 39]. In the

same Ref. [37], the relation between the particle mass m and the model parameters $z_{1,2}$ was established under the conformal normalization, and an explicit formula for the mean value of the exponential field $\langle e^{a\phi} \rangle$ was suggested by exploring a reflection relationship between the BD model and the 2D Liouville theory [40, 41]. The derivation of the results was based on special analyticity assumptions, so their verification by various checks is needed.

The spectrum of the cBD model (3.4) exhibits an extremely complicated hierarchy of particles [42]. The fundamental particle is a three-component kink. The kinks generate one-component bound states (breathers) and higher kinks, these higher kinks generate new breathers and new kinks, etc. The important simplifying fact is that, in the whole stability interval of interest $0 \le b^2 < 1$ ($0 \le \beta < 4$), the lightest particle, the 1-breather, corresponds to the analytic continuation of the only particle in the spectrum of the BD model. Since the lightest particle dominates in the thermodynamic limit $V \to \infty$, we can apply all results of Ref. [37] presented in the above paragraph, with the substitutions $b \to ib$ and $z_{1,2} \to -z_{1,2}$, also to the cBD model. In particular, the specific grand potential (3.5) takes the form

$$-\omega = \frac{m^2}{16\sqrt{3}\sin(\pi\xi/3)\sin(\pi(1+\xi)/3)} \tag{3.6a}$$

$$\xi = \frac{b^2}{2-b^2} \tag{3.6b}$$

and the mass of the lightest 1-breather reads

$$m = \frac{2\sqrt{3}\Gamma(1/3)}{\Gamma(1-\xi/3)\Gamma((1+\xi)/3)} \left[\frac{z_1 \pi \Gamma(1-b^2)}{\Gamma(b^2)} \right]^{(1+\xi)/6} \left[\frac{2 z_2 \pi \Gamma(1-b^2/4)}{\Gamma(b^2/4)} \right]^{(1+\xi)/3} \tag{3.7}$$

Note that at the collapse point $b^2 = 1$ ($\beta = 4$) the particle mass $m \to \infty$. The expected divergence of $-\omega$ (3.6) at $b^2 = 1$ is therefore caused by m^2, while the prefactor to m^2 remains finite. Such behavior is opposite to that observed in the symmetric 2D TCP [16] at the corresponding collapse point where the mass of the lightest sine-Gordon breather, m_1, is finite and $-\omega$ diverges due to the prefactor to m_1^2. The expectation value of the

exponential field is given by (see also [43])

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}a\phi}\rangle &= \left[\frac{z_2}{z_1}\frac{2^{-b^2/2}\Gamma(1+b^2)\Gamma(1-b^2/4)}{\Gamma(1-b^2)\Gamma(1+b^2/4)}\right]^{\frac{2a}{3b}}\left[\frac{m\Gamma(1-\xi/3)\Gamma((1+\xi)/3)}{2^{2/3}\sqrt{3}\Gamma(1/3)}\right]^{2a^2-ab} \\ &\quad\times\exp\left\{\int_0^\infty\frac{\mathrm{d}t}{t}\left[\frac{\sinh((2-b^2)t)\Psi(t,a)}{\sinh(3(2-b^2)t)\sinh(2t)\sinh(b^2t)}-2a^2\mathrm{e}^{-2t}\right]\right\} \qquad (3.8a)\end{aligned}$$

where

$$\begin{aligned}\Psi(t,a) &= -\sinh(2abt)\Big[\sinh((4-b^2-2ab)t)-\sinh((2-2b^2+2ab)t) \\ &\quad +\sinh((2-b^2-2ab)t)-\sinh((2-b^2+2ab)t) \\ &\quad -\sinh((2+b^2-2ab)t)\Big] \qquad (3.8b)\end{aligned}$$

The integral in (3.8a) is well defined if

$$-\frac{1}{2b}<\mathrm{Re}(a)<\frac{1}{b} \qquad (3.9)$$

Taking into account (2.12), the charge-neutrality condition in the considered Coulomb gas, $n_1=n_2/2$, results in the equality

$$z_1\langle \mathrm{e}^{\mathrm{i}b\phi}\rangle=\frac{z_2}{2}\langle \mathrm{e}^{-\mathrm{i}(b/2)\phi}\rangle \qquad (3.10)$$

Using the integral formula for the logarithm of the Gamma function [44]

$$\ln\Gamma(z)=\int_0^\infty\frac{\mathrm{d}t}{t}\mathrm{e}^{-t}\left[(z-1)+\frac{\mathrm{e}^{-(z-1)t}-1}{1-\mathrm{e}^{-t}}\right], \qquad \mathrm{Re}(z)>0 \qquad (3.11)$$

it can be readily verified that the suggested formula (3.8) is consistent with the neutrality condition (3.10).

We are now ready to derive the basic density-fugacity relationship for the asymmetric $1/-\frac{1}{2}$ Coulomb gas. The total particle number density is

$$n=z_1\frac{\partial(-\omega)}{\partial z_1}+z_2\frac{\partial(-\omega)}{\partial z_2}=(1+\xi)(-\omega) \qquad (3.12)$$

where the auxiliary parameter ξ, introduced in (3.6b), is expressible in the inverse temperature $\beta=4b^2$ as follows

$$\xi=\frac{\beta}{8-\beta} \qquad (3.13)$$

After some algebra, Eqs. (3.6) and (3.7) give the explicit density-fugacity relationship

$$\begin{aligned}\frac{n^{1-\beta/8}}{(z_1 z_2^2)^{1/3}} &= \left[\frac{\sqrt{3}}{4}\frac{\Gamma^2(1/3)}{(1-\beta/8)\pi^2}\frac{\Gamma(\xi/3)\Gamma((2-\xi)/3)}{\Gamma(1-\xi/3)\Gamma((1+\xi)/3)}\right]^{1-\beta/8} \\ &\quad\times\left[\frac{\pi\Gamma(1-\beta/4)}{\Gamma(\beta/4)}\right]^{1/3}\left[\frac{2\pi\Gamma(1-\beta/16)}{\Gamma(\beta/16)}\right]^{2/3} \qquad (3.14)\end{aligned}$$

The length constant r_0 in (2.2) was set to unity. This can be shown to imply that the fugacity product $z_1 z_2^2$ in neutral configurations of (2.6) has dimension [length]$^{-6(1-\beta/8)}$, and so Eq. (3.14) is dimensionally correct. The small-β expansion of the rhs of (3.14) reads

$$\frac{n^{1-\beta/8}}{(z_1 z_2^2)^{1/3}} = \frac{3}{2^{2/3}}\beta^{\beta/8}\exp\left\{\left[2C+\ln\left(\frac{\pi}{4}\right)\right]\frac{\beta}{8}+\left[3\psi'(2/3)-2\pi^2\right]\frac{\beta^2}{1728}+O(\beta^3)\right\} \quad (3.15)$$

where C is the Euler number and $\psi(x) = \mathrm{d}[\ln\Gamma(x)]/\mathrm{d}x$ is the psi function. The series representation of the first derivative of ψ reads

$$\psi'(x) = \sum_{j=0}^{\infty}\frac{1}{(x+j)^2} \qquad (3.16)$$

The series expansion (3.15) is checked up to the indicated order in Appendix A.1 by using a bond-renormalized Mayer expansion in density [31, 32]. For a fixed fugacity product $z_1 z_2^2$, relation (3.14) implies the expected collapse singularity of the density n:

$$\frac{n}{(z_1 z_2^2)^{2/3}} \sim \frac{\sqrt{3}}{2\pi}\left[\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\right]^2\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^{4/3}\frac{1}{(1-\beta/4)^{2/3}} \qquad \text{as } \beta\to 4^- \qquad (3.17)$$

This singularity is reproduced indirectly in Appendix B.1 by applying a (slightly modified) perfect-screening sum rule, which is another important check of the basic result (3.14).

To obtain the complete thermodynamics of the asymmetric $1/-\frac{1}{2}$ Coulomb gas, we pass from the grandcanonical to the canonical ensemble via the Legendre transformation

$$F(T;N_1,N_2) = \Omega + \mu_1 N_1 + \mu_2 N_2 \qquad (3.18)$$

where

$$\Omega = k_{\mathrm{B}}T\omega(\beta,n)V \qquad (3.19a)$$

$$-\omega(\beta,n) = \left(1-\frac{\beta}{8}\right)n \qquad (3.19b)$$

$N_1 = N/3$, $N_2 = 2N/3$ with $N = nV$ being the total particle number, and

$$\mu_\sigma(\beta, n) = k_{\rm B}T \ \ln z_\sigma(\beta, n), \qquad (\sigma = 1, 2) \tag{3.20}$$

is the chemical potential of species σ. The dimensionless specific free energy, $f = F/(Nk_{\rm B}T)$, is thence given by

$$f(\beta, n) = -\left(1 - \frac{\beta}{8}\right) + \frac{1}{3}\ln\left(z_1 z_2^2\right) \tag{3.21}$$

With the aid of the density-fugacity relationship (3.14), one has explicitly

$$\begin{aligned} f(\beta, n) \ = \ & -\left(1 - \frac{\beta}{8}\right) + \left(1 - \frac{\beta}{8}\right)\ln\left(\frac{4}{\sqrt{3}}n\right) + \left(1 - \frac{\beta}{4}\right)\ln\pi \\ & -\left(1 - \frac{\beta}{8}\right)\ln\left[\frac{\Gamma^2(1/3)}{(1-\beta/8)}\frac{\Gamma(\xi/3)\Gamma((2-\xi)/3)}{\Gamma(1-\xi/3)\Gamma((1+\xi)/3)}\right] \\ & -\frac{1}{3}\ln\left[\frac{\Gamma(1-\beta/4)}{\Gamma(\beta/4)}\right] - \frac{2}{3}\ln\left[\frac{2\Gamma(1-\beta/16)}{\Gamma(\beta/16)}\right] \end{aligned} \tag{3.22}$$

The (excess) internal energy per particle, $u^{\rm ex} = \langle E\rangle/N$, and the (excess) specific heat at constant volume per particle, $c_V^{\rm ex} = C_V^{\rm ex}/N$, are determined by elementary thermodynamics as follows

$$u^{\rm ex} = \frac{\partial}{\partial\beta} f(\beta, n) \tag{3.23a}$$

$$\frac{c_V^{\rm ex}}{k_{\rm B}} = -\beta^2 \frac{\partial^2}{\partial\beta^2} f(\beta, n) \tag{3.23b}$$

The specific heat is independent of the particle number density n, which is a specificity of the 2D pointlike Coulomb gases. The expansion of $c_V^{\rm ex}$ near the collapse point $\beta = 4$ results into the Laurent series

$$\frac{c_V^{\rm ex}}{k_{\rm B}} = \frac{1}{3(1-\beta/4)^2} - \frac{2}{3(1-\beta/4)} + O(1), \qquad \beta \to 4^- \tag{3.24}$$

The leading two terms in (3.24) are reproduced indirectly in Appendix B.2 by applying an independent-pair conjecture of Hauge and Hemmer [6].

4 Large-distance behavior of particle correlations

In a 2D integrable field theory with particle spectrum $\{\epsilon\}$, correlation functions of local operators $\mathcal{O}_a$ (a is a free parameter) are expressible as infinite convergent series over multi-particle intermediate states (see e.g. Ref. [45]). For the truncated two-point correlation functions

$$\langle \mathcal{O}_a(\mathbf{r})\mathcal{O}_{a'}(\mathbf{r}')\rangle_{\mathrm{T}} = \langle \mathcal{O}_a(\mathbf{r})\mathcal{O}_{a'}(\mathbf{r}')\rangle - \langle \mathcal{O}_a\rangle\langle \mathcal{O}_{a'}\rangle \tag{4.1}$$

the series reads

$$\begin{aligned}\langle \mathcal{O}_a(\mathbf{r})\mathcal{O}_{a'}(\mathbf{r}')\rangle_{\mathrm{T}} &= \sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\epsilon_1,\ldots,\epsilon_N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N}{(2\pi)^N}F_a(\theta_1,\ldots,\theta_N)_{\epsilon_1\cdots\epsilon_N} \\ &\quad \times^{\epsilon_N\cdots\epsilon_1}F_{a'}(\theta_N,\ldots,\theta_1)\exp\left(-|\mathbf{r}-\mathbf{r}'|\sum_{j=1}^{N}m_{\epsilon_j}\cosh\theta_j\right)\end{aligned} \tag{4.2}$$

Here, $\theta \in (-\infty,\infty)$ is the rapidity which parametrizes the energy E and the momentum p of a particle ϵ of mass m_ϵ as follows

$$E = m_\epsilon \cosh\theta, \qquad p = m_\epsilon \sinh\theta \tag{4.3}$$

and the normalization constants in the form factors $\{F_a\}$ depend on the specific form of the operator $\mathcal{O}_a$. In what follows, we will consider $\mathcal{O}_a$ to be an exponential field:

$$\mathcal{O}_a(\mathbf{r}) = \exp\left(a\phi(\mathbf{r})\right) \tag{4.4a}$$

in the case of the BD model with the action (3.3), and

$$\mathcal{O}_a(\mathbf{r}) = \exp\left(\mathrm{i}a\phi(\mathbf{r})\right) \tag{4.4b}$$

in the case of the cBD model with the action (3.4).

The form-factor representation (4.2) is particularly useful in the large-distance limit $|\mathbf{r}-\mathbf{r}'| \to \infty$. The dominant contribution in this asymptotic limit comes from a one-particle intermediate state with the minimum value of the particle mass m, at the point of the vanishing rapidity $\theta \to 0$. The consequent exponential decay $\exp(-m|\mathbf{r}-\mathbf{r}'|)$ is multiplied by a slower (inverse power law) decaying function, whose particular form

depends on the one-particle form factors. For the BD model (3.3) with the only particle ($\epsilon = 1$) in the spectrum, the one-particle form factors $F_a(\theta)_1 = F_a$ and ${}^1F_{a'}(\theta) = F^*_{a'}$ are presented for the exponential field (4.4a) in Refs. [46, 47]. The transition to the cBD model (3.4) is straightforward since, as was already mentioned in Section 3, the analytic continuation of the single particle in the BD spectrum corresponds to the lightest particle of interest in the cBD model, the 1-breather with mass m given by (3.7). In particular, after the substitutions $a \to \mathrm{i}a$ and $b \to \mathrm{i}b$ in the one-particle form-factor formulae for the BD model [46, 47], the lightest-particle form factor of the exponential field (4.4b) in the cBD model is given by

$$\frac{F_a}{\langle \mathcal{O}_a \rangle} = 4\rho \sin\left(\frac{2\pi a}{3b}\xi\right) \cos\left(\frac{\pi}{6}\left(1 + 2\xi - 4\xi\frac{a}{b}\right)\right) \tag{4.5a}$$

where

$$\begin{aligned} \rho &= \mathrm{i}\left[\frac{\sin(\pi/3)}{\sin(2\pi\xi/3)\sin(2\pi(1+\xi)/3)}\right]^{1/2} \\ &\quad \times \exp\left\{-2\int_0^\infty \frac{\mathrm{d}t}{t} \frac{\cosh(t/6)\sinh(t\xi/3)\sinh(t(1+\xi)/3)}{\sinh t \cosh(t/2)}\right\} \end{aligned} \tag{4.5b}$$

According to formulae (2.12)-(2.16) adapted to the considered asymmetric $1/-\frac{1}{2}$ 2D Coulomb gas, the pair correlation function $h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}')$ of species σ and σ' with the corresponding charges q_σ and $q_{\sigma'} \in \{1, -1/2\}$ is expressible in terms of the averages over the equivalent cBD model as follows

$$h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') = \frac{\langle \mathrm{e}^{\mathrm{i}q_\sigma b\phi(\mathbf{r})} \mathrm{e}^{\mathrm{i}q_{\sigma'} b\phi(\mathbf{r}')} \rangle_{\mathrm{T}}}{\langle \mathrm{e}^{\mathrm{i}q_\sigma b\phi} \rangle \langle \mathrm{e}^{\mathrm{i}q_{\sigma'} b\phi} \rangle} \tag{4.6}$$

Using the form-factor representation (4.2) for $\mathcal{O}_a(\mathbf{r}) = \exp(\mathrm{i}a\phi(\mathbf{r}))$, the leading contribution to $h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}')$ in the limit $|\mathbf{r} - \mathbf{r}'| \to \infty$ is determined by the lightest particle in the cBD model, the 1-breather with mass m given by (3.7) and the form factor F_a given by (4.5). Consequently,

$$h_{\sigma\sigma'}(r) \sim \frac{F_{q_\sigma b}}{\langle \mathcal{O}_{q_\sigma b} \rangle} \frac{F^*_{q_{\sigma'} b}}{\langle \mathcal{O}_{q_{\sigma'} b} \rangle} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2} \mathrm{e}^{-mr\cos\theta} \qquad \text{as } r \to \infty \tag{4.7}$$

Since

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2} \mathrm{e}^{-mr\cos\theta} = K_0(mr) \sim \left(\frac{\pi}{2mr}\right)^{1/2} \exp(-mr) \tag{4.8}$$

at asymptotically large r (K_0 is the modified Bessel function of second kind), we finally arrive at

$$h_{\sigma\sigma'}(r) \sim -\lambda_{\sigma\sigma'} \left(\frac{\pi}{2mr}\right)^{1/2} \exp(-mr) \tag{4.9}$$

Here,

$$\begin{aligned} \lambda_{\sigma\sigma'} \quad = \quad & \frac{8\sqrt{3}}{\pi} \exp\left\{ -4 \int_0^{\infty} \frac{\mathrm{d}t}{t} \frac{\cosh(t/6)\sinh(t\xi/3)\sinh(t(1+\xi)/3)}{\sinh t \cosh(t/2)} \right\} \\ & \times \frac{1}{\sin(2\pi\xi/3)\sin(2\pi(1+\xi)/3)} \sin\left(\frac{2\pi}{3}\xi q_\sigma\right) \sin\left(\frac{2\pi}{3}\xi q_{\sigma'}\right) \\ & \times \cos\left(\frac{\pi}{6}(1+2\xi-4\xi q_\sigma)\right) \cos\left(\frac{\pi}{6}(1+2\xi-4\xi q_{\sigma'})\right) \end{aligned} \tag{4.10}$$

and the mass m (3.7) is expressible in terms of the inverse Debye length for the considered Coulomb gas

$$\kappa = (\pi\beta n)^{1/2} \tag{4.11}$$

by combining Eqs. (3.6) and (3.12),

$$m = \kappa \left[\frac{2\sqrt{3}}{\pi\xi} \sin\left(\frac{\pi\xi}{3}\right) \sin\left(\frac{\pi}{3}(1+\xi)\right) \right]^{1/2} \tag{4.12}$$

We recall that $\xi = \beta/(8-\beta)$.

In the high-temperature limit, the parameters $\lambda_{\sigma\sigma'}$ (4.10) and m (4.12) of the large-distance asymptotics (4.9) have the following small-β expansions

$$\lambda_{\sigma\sigma'} \quad = \quad \beta q_\sigma q_{\sigma'} + \beta^2 q_\sigma q_{\sigma'} \left\{ \frac{1}{24} + \frac{\pi}{72\sqrt{3}} \left[6(q_\sigma + q_{\sigma'}) - 1 \right] \right\} + O(\beta^3) \tag{4.13a}$$

$$m \quad = \quad \kappa \left[1 + \frac{\pi\beta}{48\sqrt{3}} + O(\beta^2) \right] \tag{4.13b}$$

These small-β expansions are checked up to the indicated order in Appendix A.2 by using the renormalized Mayer expansion. The leading-order terms $\lambda_{\sigma\sigma'} \sim \beta q_\sigma q_{\sigma'}$ and $m \sim \kappa$ correspond to the Debye-Hückel approximation, the first corrections to this approximation are implied by the renormalized Meeron graph.

Let us introduce the charge density and particle number density combinations of the pair correlation functions

$$h_\rho(\mathbf{r},\mathbf{r}') = \sum_{\sigma,\sigma'} n_\sigma q_\sigma n_{\sigma'} q_{\sigma'} h_{\sigma\sigma'}(\mathbf{r},\mathbf{r}') \tag{4.14a}$$

$$h_n(\mathbf{r},\mathbf{r}') = \sum_{\sigma,\sigma'} n_\sigma n_{\sigma'} h_{\sigma\sigma'}(\mathbf{r},\mathbf{r}') \tag{4.14b}$$

respectively. It is evident that within the Debye-Hückel approximation the two-particle correlations are determined at large distances by the charge-charge correlation function h_ρ, while h_n is identically equal to zero. Taking into account the first correction in $\lambda_{\sigma\sigma'}$ (4.13a), the strong division between $h_\rho(r)$ and $h_n(r)$ disappears: both of them decay at large r exponentially with the same correlation length $= 1/m$, only the corresponding β-dependent prefactors differ from one another. The prefactor is of the form $\lambda_{\sigma\sigma'} = q_\sigma q_{\sigma'} \lambda(\beta)$ in the case of the symmetric 2D TCP [19, 20], and so at any β in the stability regime the large-distance asymptotics of the two-particle correlations are determined exclusively by h_ρ (h_n is related to the heavier 2-breather and therefore goes to zero faster then h_ρ). The asymmetry in the particle charges thus causes a fundamental change in the relative large-distance behavior of the charge and density correlation functions.

Since $\kappa \to \infty$ at the collapse point $\beta = 4$, the particle mass m (4.12) diverges, and $h_{\sigma\sigma'}$ given by (4.9) reduces trivially to zero. On the other hand, the mass of the lightest particle is finite (for a fixed z) at the collapse point for the symmetric 2D TCP [19], and the corresponding Ursell functions $U_{\sigma\sigma'}(\mathbf{r},\mathbf{r}') = n_\sigma n_{\sigma'} h_{\sigma\sigma'}(\mathbf{r},\mathbf{r}')$ have a nontrivial large-distance dependence. Also from this point of view the asymmetry in the particle charges has a relevant influence on the large-distance characteristics of particle correlation functions.

5 Conclusion

In this paper, we have solved exactly the 2D Coulomb gas of pointlike charged particles, with the charge asymmetry $q_1 = 1$ and $q_2 = -1/2$, via the equivalent 2D cBD field theory.

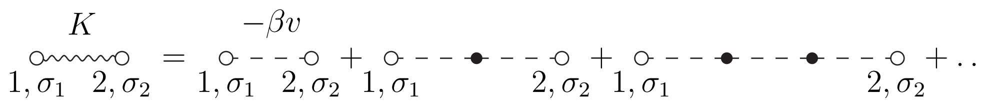
The previous calculations in the cBD model were based on special analyticity conjectures. This is why we check our results for the asymmetric Coulomb gas, in the small-β limit and close to the collapse $\beta = 4$ point. The small-β series expansions are generated using the renormalized Mayer expansion in density, which is a simplified Coulomb version of the Feynman diagrammatic perturbation technique in 2D field theories. On the other hand, the important check of the results close to the opposite collapse point, based on screening properties of the Coulomb plasma, is original and has no counterpart in the field theory.

The asymmetry in the strength of the particle charges brings two fundamental modifications in the statistical behavior of the 2D Coulomb gas in comparison with its symmetric version. Firstly, the large-distance exponential decays of the charge and density correlation functions are characterized by the same correlation length (in the symmetric Coulomb gas [19, 20], the charge correlation length is twice larger than the density one). Such behavior does not occur in the Debye-Hückel treatment of the model. Secondly, the mass of the lightest particle in the spectrum of the cBD model, which determines the thermodynamics and the asymptotics of the particle correlations, diverges at the collapse point $\beta = 4$. As a consequence, the truncated particle distributions are trivially equal to zero at $\beta = 4$. This is in contrast to the symmetric 2D Coulomb gas with finite and nonzero particle distributions at its collapse free-fermion point.

The other cases of the asymmetric Coulomb gases do not belong to the family of integrable 2D Toda field theories. On the other hand, the 2D OCP, which corresponds to the extreme charge-asymmetric case, is exactly solvable at its free-fermion point [11, 12]. The 2D OCP has a field representation [48] which resembles the one of the 2D complex Liouville model with a kind of "background" charge. Although this theory does not belong to the Toda theories, it is integrable at least at the aforementioned free-fermion point. It might be therefore useful to explore its integrability properties at an arbitrary temperature, first in the classical limit and subsequently at the quantum level.

Appendix A: Small-β expansions

Let us consider a general fluid composed of distinct species of particles $\{\sigma\}$ with the corresponding position-dependent densities $\{n(\mathbf{r},\sigma)\}$. The particles i and j interact through the pair potential $v(i,\sigma_i|j,\sigma_j)$, where vector position $\mathbf{r}_i$ is represented simply by i. The technique of the bond-renormalization in the ordinary Mayer expansion in density (for details, see Refs. [16, 31, 32]) is based on an expansion of each Mayer function in the inverse temperature β, and on a consequent series resummation of two-coordinated field circles. The renormalized K-bonds are given by



or, algebraically,

$$
\begin{aligned}
K(1,\sigma_1|2,\sigma_2) &= [-\beta v(1,\sigma_1|2,\sigma_2)] \\
&\quad + \sum_{\sigma_3} \int \mathrm{d}3\ [-\beta v(1,\sigma_1|3,\sigma_3)]\ n(3,\sigma_3)\ K(3,\sigma_3|2,\sigma_2)
\end{aligned}
\tag{A.1}
$$

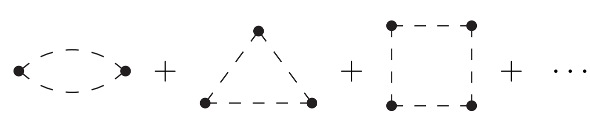
It is straightforward to verify by variation of (A.1) that it holds

$$
\frac{\delta K(1,\sigma_1|2,\sigma_2)}{\delta n(3,\sigma_3)} = K(1,\sigma_1|3,\sigma_3)K(3,\sigma_3|2,\sigma_2) \tag{A.2}
$$

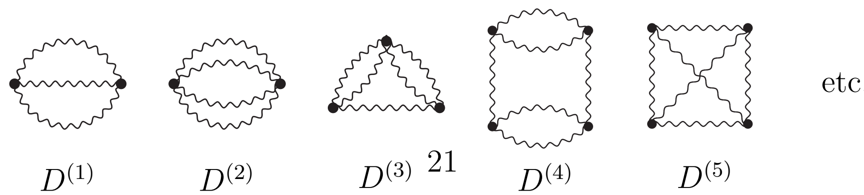
The excess Helmholtz free energy F^{ex} is expressible in the renormalized format as follows

$$-\beta F^{\mathrm{ex}}[n] =$$ $$+ \quad D^{(0)}[n] + \sum_{s=1}^{\infty} D^{(s)}[n] \tag{A.3a}$$

where

$$D^{(0)} =$$  (A.3b)

is the sum of all ring diagrams which cannot undertake the bond-renormalization procedure and



(A.3c)

are all remaining completely renormalized graphs. The free energy is the generating functional for one-body, two-body, etc. densities. The density-fugacity relationship is generated via

$$\ln\left[\frac{n(1,\sigma_1)}{z(1,\sigma_1)}\right] = \frac{\delta(-\beta F^{\mathrm{ex}})}{\delta n(1,\sigma_1)}$$

$$= \underset{1,\sigma_1}{\circ}\text{- - - -}\bullet \;+\; d^{(0)}(1,\sigma_1) + \sum_{s=1}^{\infty} d^{(s)}(1,\sigma_1) \tag{A.4a}$$

where $d^{(0)}(1,\sigma_1) = \delta D^{(0)}/\delta n(1,\sigma_1)$ can be readily obtained in the form

$$d^{(0)}(1,\sigma_1) = \frac{1}{2!}\lim_{2\to 1}\left[K(1,\sigma_1|2,\sigma_2) + \beta v(1,\sigma_1|2,\sigma_2)\right]\Big|_{\sigma_2=\sigma_1} \tag{A.4b}$$

and

$$d^{(s)}(1,\sigma_1) = \frac{\delta D^{(s)}}{\delta n(1,\sigma_1)} \tag{A.4c}$$

The direct correlation function is generated via

$$c(1,\sigma_1|2,\sigma_2) = \frac{\delta^2(-\beta F^{\mathrm{ex}})}{\delta n(1,\sigma_1)\delta n(2,\sigma_2)}$$

$$= \underset{1,\sigma_1}{\circ}\text{- - - -}\underset{2,\sigma_2}{\circ} \;+\; c^{(0)}(1,\sigma_1|2,\sigma_2) + \sum_{s=1}^{\infty} c^{(s)}(1,\sigma_1|2,\sigma_2) \tag{A.5a}$$

where $c^{(0)}(1,\sigma_1|2,\sigma_2) = \delta^2 D^{(0)}/\delta n(1,\sigma_1)\delta n(2,\sigma_2)$ corresponds to the Meeron graph,

$$c^{(0)}(1,\sigma_1|2,\sigma_2) = \underset{1,\sigma_1}{\circ}\quad\underset{2,\sigma_2}{\circ} = \frac{1}{2!}K^2(1,\sigma_1|2,\sigma_2) \tag{A.5b}$$

and

$$c^{(s)}(1,\sigma_1|2,\sigma_2) = \frac{\delta^2 D^{(s)}}{\delta n(1,\sigma_1)\delta n(2,\sigma_2)} \tag{A.5c}$$

The pair correlation function h, defined by relations (2.13) - (2.16), is related to c via the Ornstein-Zernike (OZ) equation

$$h(1,\sigma_1|2,\sigma_2) = c(1,\sigma_1|2,\sigma_2) + \sum_{\sigma_3}\int \mathrm{d}3\; c(1,\sigma_1|3,\sigma_3)\; n(3,\sigma_3)\; h(3,\sigma_3|2,\sigma_2) \tag{A.6}$$

Notice that the functional derivatives with respect to the density field generate root circles not only at obvious field-circle positions, but also on K bonds according to formula (4.2), causing their right K-K division.

Let us return to the 2D asymmetric Coulomb gas with two kinds of particles $\sigma = 1, 2$ of charges ($q_1 = 1, q_2 = -1/2$), interacting via the logarithmic interaction

$$v(i, \sigma_i | j, \sigma_j) = q_{\sigma_i} q_{\sigma_j} v(i, j) \tag{A.7a}$$

$$v(i, j) = -\ln |i - j| \tag{A.7b}$$

We consider the infinite-volume limit, characterized by homogeneous densities $n(1, \sigma) = n_\sigma$ constrained by the neutrality condition $\sum_\sigma q_\sigma n_\sigma = 0$, so that

$$n_1 = \frac{n}{3}, \qquad n_2 = \frac{2n}{3} \tag{A.8}$$

with n being the total particle density. Two-body functions are both isotropic and translationally invariant, $c(1, \sigma_1 | 2, \sigma_2) = c_{\sigma_1 \sigma_2}(|1 - 2|)$, $h(1, \sigma_1 | 2, \sigma_2) = h_{\sigma_1 \sigma_2}(|1 - 2|)$. It follows from Eq. (A.1) that the renormalized bonds exhibit the same charge-dependence as the interaction under consideration (A.7),

$$K(1, \sigma_1 | 2, \sigma_2) = q_{\sigma_1} q_{\sigma_2} K(1, 2) \tag{A.9a}$$

$$K(1, 2) = -\beta K_0(\kappa |1 - 2|) \tag{A.9b}$$

Here, K_0 is the modified Bessel function of second kind and

$$\kappa = \left[2\pi\beta \left(n_1 + \frac{n_2}{4} \right) \right]^{1/2} = \sqrt{\pi \beta n} \tag{A.10}$$

is the inverse Debye length.

A.1 Density-fugacity relationship

The small-x expansion of the modified Bessel function $K_0(x)$ reads [44]

$$K_0(x) = -\ln \left(\frac{x}{2} \right) \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j} (j!)^2} + \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j} (j!)^2} \psi(j + 1) \tag{A.11}$$

where $\psi(1) = -C$. Thus, in the uniform regime with the interaction (A.7) and the renormalized interaction (A.9), Eq. (A.4b) yields

$$d_\sigma^{(0)} = \frac{\beta q_\sigma^2}{2} \left[C + \ln \left(\frac{\kappa}{2} \right) \right], \qquad \sigma = 1, 2 \tag{A.12}$$

The first term on the rhs of (A.4a) disappears due to the charge neutrality. Consequently,

$$\ln\left(\frac{n_\sigma}{z_\sigma}\right) = \frac{\beta q_\sigma^2}{2}\left[C + \frac{1}{2}\ln\left(\frac{\pi\beta n}{4}\right)\right] + \frac{\partial}{\partial n_\sigma}\sum_{s=1}^{\infty}\frac{D^{(s)}}{V}, \qquad \sigma = 1,2 \tag{A.13}$$

where $V \to \infty$ is the volume of the system. For the dimensionless quantity of interest $n^{1-\beta/8}/(z_1 z_2^2)^{1/3}$, (A.13) gives

$$\frac{n^{1-\beta/8}}{(z_1 z_2^2)^{1/3}} = \frac{3}{2^{2/3}}\beta^{\beta/8}\exp\left\{\left[2C + \ln\left(\frac{\pi}{4}\right)\right]\frac{\beta}{8} - \left(\frac{1}{3}\frac{\partial}{\partial n_1} + \frac{2}{3}\frac{\partial}{\partial n_2}\right)\sum_{s=1}^{\infty}\frac{D^{(s)}}{V}\right\} \tag{A.14}$$

When all $\{D^{(s)}/V\}$ are set to zero, we have the Debye-Hückel approximation valid in the $\beta \to 0$ limit and, indeed, (A.14) then reproduces the leading term of the exact β-expansion (3.15). The scaling form of the renormalized interaction bonds (A.9) permits us to perform the β-classification of $D^{(s)}/V$. Let the given completely renormalized diagram $D^{(s)}$ be composed of N_s skeleton vertices and L_s bonds. Every dressed bond K brings the factor $-\beta$ and enforces the substitution $r' = \kappa r$ which manifests itself as the factor $1/\kappa^2$ for each field-circle integration $\sim \int r \mathrm{d}r$. Since there are $(N_s - 1)$ independent field-circle integrations in $D^{(s)}$ we conclude that

$$\frac{D^{(s)}(\beta)}{V} = \beta^{L_s - N_s + 1}\frac{D^{(s)}(\beta = 1)}{V} \tag{A.15}$$

In the sketch (A.3c), the only diagram which contributes to the β^2 order is $D^{(1)}$, the next diagrams $D^{(2)}, D^{(3)}, D^{(4)}$ and $D^{(5)}$ constitute the complete set of contributions to the β^3 order, etc.

We are interested in the lowest correction to the Debye-Hückel limit, so only the diagram $D^{(1)}$ has to be analyzed. The contribution of this diagram is expressible as

$$\begin{aligned}\frac{D^{(1)}(n_1, n_2)}{V} &= \frac{1}{2!3!}\left(n_1^2 - 2n_1\frac{n_2}{2^3} + \frac{n_2^2}{2^6}\right)\int \mathrm{d}^2 r \left[-\beta K_0(\kappa\mathbf{r})\right]^3 \\ &= -\frac{1}{2!3!}\beta^2\frac{(n_1 - n_2/8)^2}{n_1 + n_2/4}\int \frac{\mathrm{d}^2 r}{2\pi}K_0^3(\mathbf{r})\end{aligned} \tag{A.16}$$

To evaluate the integral in (A.16), we can make use of the 2D Fourier components of $K_0(\mathbf{r})$ and $K_0^2(\mathbf{r})$ (see Appendix of Ref. [16]) to derive

$$\int \frac{\mathrm{d}^2 r}{2\pi}K_0^3(\mathbf{r}) = \int_0^\infty \mathrm{d}k \frac{\ln\left[(k/2) + \sqrt{1 + (k/2)^2}\right]}{(1 + k^2)\sqrt{1 + (k/2)^2}} \tag{A.17}$$

After the substitution $k = 2\sin(t/2)$, one gets

$$\begin{aligned}\int \frac{\mathrm{d}^2 r}{2\pi} K_0^3(\mathbf{r}) &= \frac{1}{\sqrt{3}} \sum_{j=1}^{\infty} \frac{\sin(\pi j/3)}{j^2} \\ &= \frac{1}{72} \left[\psi'\left(\frac{1}{6}\right) + \psi'\left(\frac{1}{3}\right) - \psi'\left(\frac{2}{3}\right) - \psi'\left(\frac{5}{6}\right) \right] \end{aligned} \tag{A.18}$$

where $\psi'(x)$ is given by (3.16). Using the readily derivable relationships [44]

$$\psi'(x) = \frac{1}{4} \left[\psi'\left(\frac{x}{2}\right) + \psi'\left(\frac{x+1}{2}\right) \right] \tag{A.19a}$$

$$\psi'(x) = \frac{1}{9} \left[\psi'\left(\frac{x}{3}\right) + \psi'\left(\frac{x+1}{3}\right) + \psi'\left(\frac{x+2}{3}\right) \right] \tag{A.19b}$$

$$\psi'(x) = \psi'(x+1) + \frac{1}{x^2} \tag{A.19c}$$

valid for any x, one finds

$$\int \frac{\mathrm{d}^2 r}{2\pi} K_0^3(\mathbf{r}) = -\frac{1}{18} \left[3\psi'\left(\frac{2}{3}\right) - 2\pi^2 \right] \tag{A.20}$$

The consideration of (A.20) in (A.16) leads to

$$-\left(\frac{1}{3} \frac{\partial}{\partial n_1} + \frac{2}{3} \frac{\partial}{\partial n_2} \right) \frac{D^{(1)}}{V} = \left[3\psi'\left(\frac{2}{3}\right) - 2\pi^2 \right] \frac{\beta^2}{1728} \tag{A.21}$$

where (A.8) was taken into account. Finally, inserting (A.21) into (A.14), the term of the order β^2 is reproduced exactly in the exponential of the small-β expansion (3.15), generated from the exact density-fugacity relationship (3.14).

A.2 Large-distance asymptotics of particle correlations

In the Fourier picture, the OZ equation (A.6) reads

$$\hat{h}_{\sigma\sigma'}(k) = \hat{c}_{\sigma\sigma'}(k) + 2\pi \sum_{\sigma''=1,2} \hat{c}_{\sigma\sigma''}(k) n_{\sigma''} \hat{h}_{\sigma''\sigma'}(k) \tag{A.22}$$

It follows from (A.5) that, at the lowest order in β (Debye-Hückel approximation),

$$c_{\sigma\sigma'}(r) = \beta q_\sigma q_{\sigma'} \ln r \tag{A.23a}$$

$$\hat{c}_{\sigma\sigma'}(k) = -\beta q_\sigma q_{\sigma'} \frac{1}{k^2} \tag{A.23b}$$

Consequently, from (A.22),

$$\hat{h}_{\sigma\sigma'}(k) = -\beta q_\sigma q_{\sigma'} \frac{1}{k^2+\kappa^2} \tag{A.24a}$$

$$h_{\sigma\sigma'}(r) = -\beta q_\sigma q_{\sigma'} K_0(\kappa r) \tag{A.24b}$$

and the asymptotic form of $h_{\sigma\sigma'}(r)$ is

$$h_{\sigma\sigma'}(r) \sim -\beta q_\sigma q_{\sigma'} \left(\frac{\pi}{2\kappa r}\right)^{1/2} \exp(-\kappa r) \tag{A.25}$$

reproducing thus the expected result (4.9) with $\lambda_{\sigma\sigma'} = \beta q_\sigma q_{\sigma'} + O(\beta^2)$ (4.13a) and $m = \kappa[1+O(\beta)]$ (4.13b).

With the aid of Eq. (A.5), the direct correlation function is determined up to the β^2 order as follows

$$c_{\sigma\sigma'}(r) = \beta q_\sigma q_{\sigma'} \ln r + \frac{\beta^2}{2} q_\sigma^2 q_{\sigma'}^2 K_0^2(\kappa r) \tag{A.26}$$

Taking $\kappa^{-1} = 1/\sqrt{\pi\beta n}$ as the unit of length, one has in the Fourier space

$$\hat{c}_{\sigma\sigma'}(k) = -\beta q_\sigma q_{\sigma'} \frac{1}{k^2} + \frac{\beta^2}{2} q_\sigma^2 q_{\sigma'}^2 \frac{\ln\left[(k/2)+\sqrt{1+(k/2)^2}\right]}{k\,\sqrt{1+(k/2)^2}} \tag{A.27}$$

The OZ equation (A.22) implies

$$\hat{h}_{11}(k) = \frac{1}{d}\left\{-\beta + \frac{\beta^2(3+4k^2)}{4k\sqrt{4+k^2}} \ln\left[\frac{k+\sqrt{4+k^2}}{2}\right]\right\} \tag{A.28a}$$

$$\hat{h}_{12}(k) = \frac{1}{d}\left\{\frac{\beta}{2} + \frac{\beta^2 k}{4\sqrt{4+k^2}} \ln\left[\frac{k+\sqrt{4+k^2}}{2}\right]\right\} \tag{A.28b}$$

$$\hat{h}_{22}(k) = \frac{1}{d}\left\{-\frac{\beta}{4} + \frac{\beta^2(6+k^2)}{16k\sqrt{4+k^2}} \ln\left[\frac{k+\sqrt{4+k^2}}{2}\right]\right\} \tag{A.28c}$$

where the denominator is

$$d = 1 + k^2 - \frac{\beta(2+3k^2)}{4k\sqrt{4+k^2}} \ln\left[\frac{k+\sqrt{4+k^2}}{2}\right] \tag{A.29}$$

The asymptotic behavior of $h_{\sigma\sigma'}(r)$ is governed by the poles of $\hat{h}_{\sigma\sigma'}(k)$ closest to the real axis. When $\beta \to 0$, there poles are at $k = \pm \mathrm{i}$ (or $k^2 = -1$). In close analogy with Ref. [19], to find the β-correction of these poles, it is sufficient to expand the β-dependent part

of the denominator d around say $k = \mathrm{i}$ up to the first order in $(k - \mathrm{i})$:

$$\begin{aligned} d &= 1 + k^2 + \frac{\pi\beta}{24\sqrt{3}} - \frac{\mathrm{i}\beta}{108}(9 + 8\sqrt{3}\pi)(k - \mathrm{i}) + \cdots \\ &= (k^2 + m^2)\left[1 - \frac{\mathrm{i}\beta}{108}\frac{(9 + 8\sqrt{3}\pi)(k - \mathrm{i})}{k^2 + m^2} + \cdots\right] \\ &= (k^2 + m^2)\left[1 - \beta\left(\frac{1}{24} + \frac{\pi}{9\sqrt{3}}\right) + \cdots\right] \end{aligned} \tag{A.30}$$

where

$$m = 1 + \frac{\pi\beta}{48\sqrt{3}} + O(\beta^2) \tag{A.31}$$

The prefactors $\lambda_{\sigma\sigma'}$, defined by

$$\hat{h}_{\sigma\sigma'}(k) = -\lambda_{\sigma\sigma'}\frac{1}{k^2 + m^2} \tag{A.32}$$

are deducible from (A.28) (with numerators evaluated at $k^2 = -1$) and (A.30),

$$\lambda_{11} = \beta + \beta^2\left(\frac{1}{24} + \frac{11\pi}{72\sqrt{3}}\right) + O(\beta^3) \tag{A.33a}$$

$$\lambda_{12} = -\frac{\beta}{2} - \frac{\beta^2}{2}\left(\frac{1}{24} + \frac{\pi}{36\sqrt{3}}\right) + O(\beta^3) \tag{A.33b}$$

$$\lambda_{22} = \frac{\beta}{4} + \frac{\beta^2}{4}\left(\frac{1}{24} - \frac{7\pi}{72\sqrt{3}}\right) + O(\beta^3) \tag{A.33c}$$

The large-distance behavior of $h_{\sigma\sigma'}(r)$ is of type (4.9): the prefactors $\lambda_{\sigma\sigma'}$ (A.33) and the parameter m (A.31) have the small-β expansions (4.13a) and (4.13b), respectively, confirming thus the predictions of the form-factor method.

Appendix B: Collapse point

B.1 Density-fugacity relationship

Based on the analysis in Section 2 applied to our specific $Q = 2$ case, close to the collapse point $\beta = 4$ ($b = 1$), the two-body densities behave like

$$n_{12}(r) \sim z_1 z_2 \langle \mathrm{e}^{\mathrm{i}\phi/2}\rangle|_{b\to 1} r^{-\beta/2} \tag{B.1a}$$

$$n_{11}(r) \propto r^1 \tag{B.1b}$$

$$n_{22}(r) \propto r^{-1} \tag{B.1c}$$

in the short-distance limit $r \to 0$. The average in (B.1a) is taken over the cBD action (3.4) with $b \to 1$.

The one-body and two-body densities in a general Coulomb system satisfy the electroneutrality sum rule [33]

$$-q_\sigma n_\sigma = \sum_{\sigma'} q_{\sigma'} \int \mathrm{d}^d r \; n_{\sigma\sigma'}(r) \tag{B.2}$$

where σ numerates the charged species. For our asymmetric $1/-\frac{1}{2}$ 2D Coulomb gas, one has in particular

$$-\frac{n}{3} = \int \mathrm{d}^2 r \; n_{11}(r) - \frac{1}{2}\int \mathrm{d}^2 r \; n_{12}(r) \tag{B.3a}$$

$$\frac{n}{3} = \int \mathrm{d}^2 r \; n_{12}(r) - \frac{1}{2}\int \mathrm{d}^2 r \; n_{22}(r) \tag{B.3b}$$

It follows from (3.17) that, for fixed z_1 and z_2, the density n exhibits the singularity of type $(1-\beta/4)^{-2/3}$ as $\beta \to 4^-$. This singularity originates in (B.3) as a result of the short-distance integration over the two-body densities $n_{12}(r) \propto r^{-\beta/2}$; neither $n_{11} \propto r^1$ nor $n_{22}(r) \propto r^{-1}$ give a diverging contribution as $\beta \to 4^-$ after being integrated out over short distances r. The problem with Eqs. (B.3) is that they imply the dependences of n on the relevant (diverging) integral $\int \mathrm{d}^2 r n_{12}(r)$ with two different prefactors. We have to admit that the collapse mechanism alters the form of the sum rules (B.3). The particles can be divided into three basic groups: one third of particles is of type 1 (charge $q_1 = +1$ and density $n_1 = 1/3$), one third of particles is of type 2 (charge $q_2 = -1/2$ and density $n_2 = 1/3$) and the remaining third of particles of type $2'$ (charge $q_{2'} = -1/2$ and density $n_{2'} = 1/3$) is excluded from the collapse phenomenon (they only feel the charge $+1/2$ of each collapsed pair of 1,2-particles and do not enter into diverging screening integrals). Under this assumption, close to $\beta = 4$, the electroneutrality sum rules (B.2) are modified as follows

$$-q_{2'}n_{2'} - q_\sigma n_\sigma \sim q_{\sigma^*} \int \mathrm{d}^2 r \; n_{\sigma\sigma^*}(r), \qquad \sigma = 1,2 \tag{B.4}$$

where $1^* = 2$ and $2^* = 1$. The couple of Eqs. (B.4) is consistent and implies the only

singular relationship

$$n \sim 3 \int \mathrm{d}^2 r \; n_{12}(r) \qquad \text{as } \beta \to 4^- \tag{B.5}$$

We now put the short-distance expansion of $n_{12}(r)$, Eq. (B.1a), into the integral on the rhs of (B.5) cut at some finite $|r| = L$ (L is a length over which the Coulomb interaction is screened) and obtain

$$n \sim z_1 z_2 \langle \mathrm{e}^{\mathrm{i}\phi/2} \rangle |_{b\to 1} \frac{3\pi}{1-\beta/4} \qquad \text{as } \beta \to 4^- \tag{B.6}$$

To evaluate the mean value of the exponential field, we apply the conjectured formula (3.8) for $a = 1/2$, $b \to 1$ and $\xi = 1$. Consequently,

$$\begin{aligned} \langle \mathrm{e}^{\mathrm{i}\phi/2} \rangle |_{b\to 1} \;\; &\sim \;\; \sqrt{2} \left[\frac{z_2}{z_1} \frac{\Gamma(3/4)}{\Gamma(1/4)} \left(1 - \frac{\beta}{4} \right) \right]^{1/3} \\ & \qquad \times \exp \left\{ \int_0^\infty \frac{\mathrm{d}t}{2t} \left[\frac{1}{\cosh t (2\cosh t - 1)} - \mathrm{e}^{-2t} \right] \right\} \end{aligned} \tag{B.7}$$

where we have applied $\Gamma(x+1) = x\Gamma(x)$. It is simple to show by using the integral representation of the logarithm of the Gamma function (3.11) that

$$\exp \left\{ \int_0^\infty \frac{\mathrm{d}t}{2t} \left[\frac{1}{\cosh t (2\cosh t - 1)} - \mathrm{e}^{-2t} \right] \right\} = \frac{1}{(2\pi)^2} \left(\frac{2}{3} \right)^{1/2} \frac{\Gamma(3/4)}{\Gamma(1/4)} \left[\Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right) \right]^2 \tag{B.8}$$

Eqs. (B.6) - (B.8) reproduce exactly the collapse singularity (3.17) deduced from the exact density-fugacity relationship (3.14).

B.2 Thermodynamics

Although the thermodynamics of the system close to the collapse point is derivable directly from the collapse $n - z$ relationship discussed in the previous subsection, there exists another simpler derivation in the spirit of an independent-pair approximation by Hauge and Hemmer [6]. As was already mentioned, close to the collapse point, from the total number of N particles, $N/3$ particles of type 1 and $N/3$ particles of type 2 form pairs. Their statistical weights contribute dominantly to the configuration integral

$$Q \propto \left[\int_0^L \mathrm{d}^2 r \; r^{-\beta/2} \right]^{N/3} = \left[\frac{\pi}{1-\beta/4} L^{2(1-\beta/4)} \right]^{N/3} \qquad \text{as } \beta \to 4^- \tag{B.9}$$

Thence

$$\frac{c_V^{\rm ex}}{k_{\rm B}} = \beta^2 \frac{\partial^2}{\partial \beta^2} \left(\frac{1}{N} \ln Q \right) = \frac{1}{3(1-\beta/4)^2} - \frac{2}{3(1-\beta/4)} + O(1) \tag{B.10}$$

in full agreement with the Laurent series (3.24).

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